

The Velocity of Energy Transport for Normal Modes Near Total Internal Reflection

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The problem of the velocity of propagation of the various normal modes of a spherical electromagnetic quasi-monochromatic wave in a transparent layer near total reflection is examined, and a formula giving the velocity of the energy transport as a function of the eigenvalues of the modal equation is derived.

It is well known that in a very dispersive medium — where the higher derivatives of the refractive index with respect to frequency cannot be omitted — the character of the velocity of energy transport of an EM pulse becomes ambiguous in the sense that this velocity depends on the spectrum of the transmitted signal. This ambiguity is a characteristic of non-linearity and cannot be removed by a suitable re-definition of the velocity of energy transport^{1, 2}; it can, however, be minimized by the use of quasi-monochromatic waves (see below).

The problem is relevant in a number of applications involving geophysical diagnostics or remote sensing via leaky channels where a precise determination of the arrival time of a disturbance is required. Although the analysis given below is valid for a great variety of leaky wave guides, for the sake of definiteness we will consider here, as an example, a homogeneous dielectric non-dispersive layer supported by a perfectly conducting interface. The layer will assume a thickness h and its refractive index is taken $n \cong 1 + \Delta n$ where $|\Delta n| \ll 1$. Both transmitter and receiver are located inside the slab. The transmitter sends a quasi-monochromatic spherical wave. It is required to calculate the velocity of energy transport of the far field.

The received field contains 3 distinct groups of energetic entities³: partially reflected normal modes (decaying in amplitude exponentially with distance r), totally reflected normal modes (decaying cylindrically with distance), and the lateral wave (which decays^{3, 4} as $1/r$ or $1/r^2$). It is obvious, therefore,

that for the far field, *only* the totally reflected normal modes should be taken into account.

We start as usual⁴ from the investigation of the roots of the pertinent modal equation. For parallel polarization, e. g., this equation assumes the form:

$$\tan Z_l = \frac{-Z_l}{\sqrt{a^2 - Z_l^2}} \quad (1)$$

where $a = K h \sqrt{2 \Delta n}$ and $K = \omega/c$ is the wave number in vacuum. $Z_l = X_l + j Y_l$ is the (generally complex) solution of (1) for the l -th (in generally partially reflected) quasi-monochromatic normal mode ($l = 1, 2, 3, \dots$).

In our case (1) will be considered mainly for $Z_l \equiv X_l = \text{real}$ which happens for

$$K h \sqrt{2 \Delta n} \geq (2l - 1) \pi/2 \quad (2)$$

for the l -th mode. Our intention is to calculate the velocity with which the energy carried along by the individual normal modes under the condition (2) is propagating between the transmitter and the receiver.

The expression for the wave number of the l -th normal mode is³

$$K_l(\omega) = K \sqrt{1 - \frac{Z_l^2}{(K h)^2}}$$

and for a partially reflected mode is of course complex.

For a totally reflected mode, i. e., for

$$\omega_i \geq \omega_{\min} = \frac{c(2l-1)\pi}{2h\sqrt{2\Delta n}},$$

$$K_l(\omega) = \frac{\omega}{c} \sqrt{1 - \frac{c^2 X_l^2}{h^2 \omega^2}} = \text{real}.$$

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¹ L. A. WAINSTEIN, Group Velocity of Damped Waves, Sov. Phys. Tech. Phys. 2, 2420 [1957].

² L. BRILLOUIN, Wave Propagation and Group Velocity, Academic Press, London 1960.

³ L. BREKHOVSKIKH, Waves in Layered Media, Academic Press, London 1960.

⁴ J. S. NICOLIS and S. HOODBOY, The Behavior of the First Normal Modes and the Lateral Wave in a Weak Dielectric Layer, Radio Sci. 3 (No. 10), 1035 [1968].



Figures 1 and 2 shows the variations of Y_l and X_l as functions of ω (for $h\sqrt{2An}$ fixed) before and after total reflection for $l=1, 2, 3$. We see that near the region of transition X_l and Y_l are, both, rapidly varying functions of ω , while far from the critical

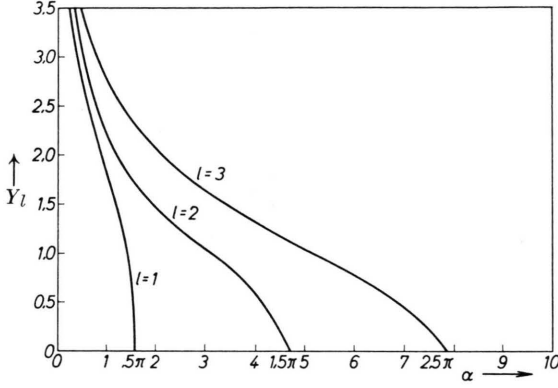


Fig. 1. Y_l as a function of a .

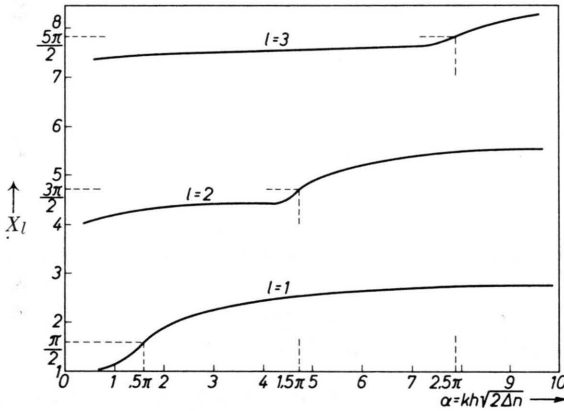


Fig. 2. X_l as a function of a .

point the change rather slowly. This suggests that higher order derivatives of $K_l(\omega)$ with respect to ω cannot inadvertently be neglected near total reflection; so the usual formula for the group velocity

$$v_{gr} \cong \frac{\partial \omega}{\partial K_l(\omega)}$$

cannot be applied near the transition region.

Our propagation scheme then is equivalent, as far as dispersion is concerned, to an unbounded

medium with a relative dielectric constant

$$\epsilon = 1 - \frac{c^2}{h^2} \frac{X_l^2}{\omega^2},$$

which (near total reflection) varies rapidly with ω .

We can extend BRILLOUIN's analysis² and include higher order derivatives $\partial^2 \epsilon / \partial \omega^2$, $\partial^3 \epsilon / \partial \omega^3$ in order to calculate the equivalent dielectric constant ϵ_t of our scheme and the corresponding velocity of energy transport v_t . The quasi-monochromatic wave is represented as the sum of two neighboring carriers

$$\omega_1 = \omega + \delta, \quad \omega_2 = \omega - \delta (\delta \ll \omega)$$

of equal amplitude, say $A/2$. In carrying out his analysis, Brillouin did *not* proceed beyond $\partial \epsilon / \partial \omega$ and as a result of this approximation, the parameter δ did not appear in his final result for v_t , which coincides with v_{gr} . In our case, however, δ , the spectral width of the quasi-monochromatic wave, is retained in the final expression for v_t (see below).

Now δ cannot physically be reduced as much as we please. In fact, $2\delta_{\min}$ is just equal to the (statistically determined) average (3 db) value of the natural line breath radiated by a single dipole oscillator as a result of the action of the self-force. The ratio $2\delta/\omega^2$ is independent of ω , a universal constant, and equals

$$\frac{2}{3} \frac{r_0}{c} \cong \frac{2 \cdot 10^{-23}}{3} \text{ s} \quad (3)$$

where r_0 is the so-called classical electron radius.

The electrical energy stored per unit time and volume in the medium from a quasi-monochromatic wave with line breath 2δ is

$$W = \frac{1}{4\pi} \int_0^{\pi/2\delta} \bar{E} \cdot \frac{d\bar{D}}{dt} = W_0 + W_1(\delta) \quad (4)$$

where W_0 is the propagating portion of this energy and $W_1(\delta)$ stands for the increase of the kinetic energy of the medium per unit volume above its "ground state".

$|\bar{E}|$ = intensity of the electric field

$$= \frac{A}{2} (\cos \omega_1 t - \cos \omega_2 t) = -A \sin(\delta t) \sin \omega t$$

and

$$\left| \frac{d\bar{D}}{dt} \right| = -\frac{A}{2} (\epsilon_1 \omega_1 \sin \omega_1 t - \epsilon_2 \omega_2 \sin \omega_2 t).$$

Following Brillouin we expand

$$\epsilon_1 \omega_1 = \epsilon \omega + \delta \frac{\partial(\epsilon \omega)}{\partial \omega} + \frac{\delta^2}{2} \frac{\partial^2(\epsilon \omega)}{\partial \omega^2} + \frac{\delta^3}{6} \frac{\partial^3(\epsilon \omega)}{\partial \omega^3} + \dots \quad (5)$$

and

$$\epsilon_2 \omega_2 = \epsilon \omega - \delta \frac{\partial(\epsilon \omega)}{\partial \omega} + \frac{\delta^2}{2} \frac{\partial^2(\epsilon \omega)}{\partial \omega^2} - \frac{\delta^3}{6} \frac{\partial^3(\epsilon \omega)}{\partial \omega^3} + \dots$$

We keep two more terms involving $\partial^2(\epsilon \omega)/\partial \omega^2$ and $\partial^3(\epsilon \omega)/\partial \omega^3$.
So (5) becomes:

$$\begin{aligned} W = & \frac{A^2}{4\pi} \left\{ \epsilon \omega + \frac{\delta^2}{2} \frac{\partial^2(\epsilon \omega)}{\partial \omega^2} \right\} \int_0^{\pi/2\delta} \sin^2(\delta t) \cos(\omega t) \sin(\omega t) dt \\ & + \frac{A^2}{4\pi} \left\{ \delta \frac{\partial(\epsilon \omega)}{\partial \omega} + \frac{\delta^3}{6} \frac{\partial^3(\epsilon \omega)}{\partial \omega^3} \right\} \int_0^{\pi/2\delta} \sin^2(\omega t) \cos(\delta t) \sin(\delta t) dt. \end{aligned} \quad (6)$$

The first integral gives:

$$\begin{aligned} I_1 = \int_0^{\pi/2\delta} \sin^2(\delta t) \cos \omega t \sin \omega t dt = & -\frac{1}{8\omega} \left\{ \cos \frac{\omega \pi}{\delta} - 1 \right\} \\ & + \frac{1}{16(\omega + \delta)} \left\{ \cos \left(\frac{\omega + \delta}{\delta} \right) \pi - 1 \right\} + \frac{1}{16(\omega - \delta)} \left\{ \cos \left(\frac{\omega - \delta}{\delta} \right) \pi - 1 \right\}. \end{aligned}$$

Expanding $\frac{1}{\omega + \delta}$ and $\frac{1}{\omega - \delta}$ in the *amplitude* terms and retaining only first order terms which gives an accuracy of the order of $(\delta/\omega)^2$ yields:

$$I_1 = -\frac{1}{4\omega} \cos \left(\frac{\omega \pi}{\delta} \right).$$

The second integral is evaluated in a similar manner and gives

$$I_2 = \int_0^{\pi/2\delta} \sin^2(\omega t) \cos(\delta t) \sin(\delta t) dt = 1/4 \delta.$$

Substituting in (6) and taking into account that

$$\frac{\partial(\epsilon \omega)}{\partial \omega} = \epsilon + \omega \frac{\partial \epsilon}{\partial \omega}, \quad \frac{\partial^2(\epsilon \omega)}{\partial \omega^2} = 2 \frac{\partial \epsilon}{\partial \omega} + \omega \frac{\partial^2 \epsilon}{\partial \omega^2} \quad \text{and} \quad \frac{\partial^3(\epsilon \omega)}{\partial \omega^3} = 3 \frac{\partial^2 \epsilon}{\partial \omega^2} + \omega \frac{\partial^3 \epsilon}{\partial \omega^3}$$

we finally get:

$$W = \frac{A^2}{16\pi} \left\{ \epsilon \left(1 - \cos \frac{\omega \pi}{\delta} \right) + \frac{\partial \epsilon}{\partial \omega} \left(\omega - \frac{\delta^2}{\omega} \cos \frac{\omega \pi}{\delta} \right) + \frac{\partial^2 \epsilon}{\partial \omega^2} \left(\frac{\delta^2}{2} - \frac{\delta^2}{2} \cos \frac{\omega \pi}{\delta} \right) + \frac{\partial^3 \epsilon}{\partial \omega^3} \frac{\delta^2 \omega}{6} \right\}.$$

In the above expression we have to remember that δ is a statistically derived average value so that all cos-terms must be taken at their average values, i. e.:

$$W = \frac{A^2}{16\pi} \left(\epsilon + \omega \frac{\partial \epsilon}{\partial \omega} + \frac{\delta^2}{2} \frac{\partial^2 \epsilon}{\partial \omega^2} + \frac{\delta^2}{6} \omega \frac{\partial^3 \epsilon}{\partial \omega^3} \right) = \frac{A^2}{16\pi} \left\{ \epsilon + \omega \frac{\partial \epsilon}{\partial \omega} + \frac{\delta^2}{6} \left[3 \frac{\partial^2 \epsilon}{\partial \omega^2} + \omega \frac{\partial^3 \epsilon}{\partial \omega^3} \right] \right\}.$$

To this electric power density one must add the magnetic power density (which for a plane wave equals $A_2^2/16\pi$), in order to get the total power density transferred to the medium. So, if we denote by ϵ_t an equivalent dielectric constant of the medium, we have

$$W_{\text{tot}} \cong \frac{A^2}{8\pi} \epsilon_t \quad [W_1(\delta) \text{ now acquires its minimum value}]$$

where

$$\epsilon_t \cong \epsilon + \frac{\omega}{2} \frac{\partial \epsilon}{\partial \omega} + \frac{\delta^2}{12} \left[3 \frac{\partial^2 \epsilon}{\partial \omega^2} + \omega \frac{\partial^3 \epsilon}{\partial \omega^3} \right]. \quad (7)$$

By omitting the higher order derivatives in the above expression we refine Brillouin's original result for a slightly dispersive medium. The velocity of energy transport will be given now by:

$$v_t = \frac{\epsilon v_{\text{ph}}}{\epsilon_t} \cong \frac{v_{\text{ph}}}{1 + \frac{\omega}{2\epsilon} \frac{\partial \epsilon}{\partial \omega} + \frac{\omega^4}{48\epsilon} \left(\frac{2\delta}{\omega^2} \right)^2 \left[3 \frac{\partial^2 \epsilon}{\partial \omega^2} + \omega \frac{\partial^3 \epsilon}{\partial \omega^3} \right]} \quad (8)$$

where the universal constant $2\delta/\omega^2$ appears explicitly. Examine for instance the second term in brackets. The factor

$$\frac{\omega^5}{48\epsilon} \left(\frac{2\delta}{\omega^2} \right)^2 \cong \frac{\omega^5}{48\epsilon} 10^{-46}$$

is not necessarily small: for a frequency of the order of a few GHz and ϵ always $\lesssim 1$ this factor may be of the order of one. Such frequencies can correspond to lower order modes in the vicinity of total reflection for say a thin layer of the order of a wavelength ($K h \sim 1$) where the refractive index is of the order of ~ 1.5 (millimeter waves in plexiglas layers for instance). In another interesting case, that of a typical atmospheric layer, ($h \sim 100$ m, $\Delta n \sim 10^{-5}$) the total reflection frequencies are, for the lower order modes, in the region of a few hundred MHz and so the above factor for those modes is small.

Taking into account that

$$v_{ph} = \frac{\omega}{K_l(\omega)} = \frac{K h c}{\sqrt{(K h)^2 - X_l^2}}$$

we get after some manipulation from (8) the velocity of energy transport for the l -th normal mode near total internal reflection as a function of the corresponding eigenvalues of the modal equation:

$$v_t(l) = \frac{\omega h}{\sqrt{\left(\frac{\omega}{c} h\right)^2 - X_l^2} + \frac{6 X_l \left(X_l - \omega \frac{\partial X_l}{\partial \omega}\right) + \delta^2 \left[\frac{3}{\omega^2} \left(X_l - \omega \frac{\partial X_l}{\partial \omega}\right)^2 + 3 X_l \frac{\partial^2 X_l}{\partial \omega^2} - 3 \omega \frac{\partial X_l}{\partial \omega} \frac{\partial^2 X_l}{\partial \omega^2} - \omega X_l \frac{\partial^3 X_l}{\partial \omega^3}\right]}}{6 \sqrt{\left(\frac{\omega}{c} h\right)^2 - X_l^2}} \quad (9)$$

Once the velocity of the l -th mode is known, the velocity with which the ensemble of the totally reflected normal modes (i. e. the totally reflected portion of the transmitted wave) propagates can be calculated as the power-average

$$v_{av} = \frac{\sum_l v_t(l) |\psi(l)|^2}{\sum_l |\psi(l)|^2}$$

where $|\psi(l)|$ is the amplitude of the l -th totally reflected normal mode^{3, 4}, at the receiving place. This expression can be evaluated numerically — in the absence of a close analytical form for (9). The number of totally reflected normal modes is given from

$$K h \sqrt{2 \Delta n} \geq (2l-1) \frac{\pi}{2} \quad \text{or} \quad l \leq \frac{2 h \sqrt{2 \Delta n}}{\lambda} + \frac{1}{2}$$

where λ is the vacuum wavelength. A suitable choice of the operating frequency, for a given h and Δn , limits drastically in some cases the number of totally reflected normal modes ($l = 1 \div 10$). This can be done for example in underwater L. F. propagation experiments.